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ON A MALKIN — MASSERA THEOREM

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The results obtained in [1, 2] are complemented by an assertion on asymptotic stability uniform with respect to $\{t_0, x_0\}$, and also are extended to the problems of asymptotic stability with respect to a part of the variables and of optimal stabilization with respect to a part of the variables.

1. Let there be given a system of differential equations of perturbed motion

$$\mathbf{x}^{\cdot} = \mathbf{X}(t, \mathbf{x}),\ \mathbf{X}(t, 0) \equiv 0, \quad \mathbf{x} \in \mathbf{R}^n \tag{1.1}$$

whose right hand sides are continuous in domain

$$t \geqslant 0, \quad \|\mathbf{x}\| \leqslant H > 0 \tag{1.2}$$

and satisfy therein the uniqueness conditions for the solution. Malkin [1] showed that if two positive-definite functions $V(t, \mathbf{x})$ and $W(t, \mathbf{x})$ exist for system (1.1), the first of which admits of an infinitesimal upper bound and whose derivative $V^{\cdot}(t, \mathbf{x})$ relative to system (1.1) satisfies the condition

$$V^{\cdot}(t, \mathbf{x}) + W(t, \mathbf{x})_{\lambda \leqslant \|\mathbf{x}\| \leqslant \mu} \rightrightarrows 0 \text{ as } t \to \infty \tag{1.3}$$

for any two numbers λ and μ such that $0 < \lambda < \mu < H$, then the unperturbed motion $\mathbf{x} = 0$ is stable. Massera [2], analyzing this result, proved that when the hypotheses of Malkin's theorem [1] are fulfilled, equiasymptotic stability (also called [3] asymptotic stability uniform in $\mathbf{x}_0$) obtains.

It is proved below that under the fulfilment of Malkin's hypotheses [1] asymptotic stability uniform in $\{t_0, \mathbf{x}_0\}$ obtains; this result is then used in application to the problem of stability relative to a part of the variables [4].

We prepresent vecotor $\mathbf{x}$ as

$$\mathbf{x} = (y_1, \ldots, y_m, z_1, \ldots, z_p),\ m > 0,\ p \geqslant 0,\ n = m + p,$$

and we assume that: a) in the domain

$$t \geqslant 0,\ \|\mathbf{y}\| \leqslant H > 0, \quad \|\mathbf{z}\| < +\infty \tag{1.4}$$

the right hand sides of system (1.1) are continuous and satisfy the uniqueness conditions for the solution $\mathbf{x} = \mathbf{x}(t; t_0, \mathbf{x}_0)$ defined by the initial conditions $\mathbf{x}(t_0; t_0, \mathbf{x}_0) = \mathbf{x}_0$; b) the solutions of system (1.1) are $\mathbf{z}$-continuable. In addition, we assume that for any $T > 0$ there exists $L(T) > 0$ such that the condition

$$\|\mathbf{X}(t, \mathbf{x}') - \mathbf{X}(t, \mathbf{x}'')\| \leqslant L \|\mathbf{x}' - \mathbf{x}''\| \tag{1.5}$$

is fulfilled in the domain $0 \leqslant t \leqslant T, \|\mathbf{x}\| \leqslant H$

T h e o r e m 1. Assume that functions $V(t, \mathbf{x})$ and $W(t, \mathbf{x})$ exist, satisfying in domain (1.4) the inequalities

$$a\,(\|\,\mathbf{y}\,\|) \leqslant V\,(t,\,\mathbf{x}) \leqslant b\,(\|\,\mathbf{y}\,\|) \tag{1.6}$$
$$W\,(t,\,\mathbf{x}) \geqslant c\,(\|\,\mathbf{y}\,\|) \tag{1.7}$$

where $a\,(r)$, $b\,(r)$ and $c\,(r)$ are continuous functions increasing monotonically when $r \in [0,\,H]$ and vanishing when $r = 0$; also, let the condition

$$V^{\cdot}\,(t,\,\mathbf{x}) + W\,(t,\,\mathbf{x})_{\lambda \leqslant \|\mathbf{y}\| \leqslant \mu,\ 0 \leqslant \|\mathbf{z}\| < \infty} \rightrightarrows 0 \qquad \text{as } t \to \infty \tag{1.8}$$

be fulfilled for any λ and μ such that $0 < \lambda < \mu < H$. Then motion $\mathbf{x} = 0$ is asymptotically $\mathbf{y}$-stable uniformly in $\{t_0,\,\mathbf{x}_0\}$.

P r o o f. 1) Let us show that motion $\mathbf{x} = 0$ is $\mathbf{y}$-stable uniformly in t_0 [1]. Let $\varepsilon \in (0,\,H)$ be given. According to (1.8) and (1.7), for numbers $\lambda = b^{-1}\,(a\,(\varepsilon))$ (b^{-1} is the function inverse to b) and $\mu = \varepsilon$ we can find $T\,(\varepsilon) > 0$ such that the inequality $V^{\cdot}\,(t,\,\mathbf{x}) < 0$ is fulfilled in domain $b^{-1}\,(a\,(\varepsilon)) \leqslant \|\,\mathbf{y}\,\| \leqslant \varepsilon$ for all $t \geqslant T\,(\varepsilon)$. By virtue of (1.5) the solutions depend continuously on the initial conditions and, consequently [5 — 7], we can choose $\delta\,(\varepsilon,\,T\,(\varepsilon)) = \delta\,(\varepsilon)$, $0 < \delta\,(\varepsilon) < b^{-1}\,(a\,(\varepsilon))$, such from $\|\,\mathbf{x}_0\,\| < \delta$, $t_0 \in [0,\,T)$ follows $\|\,\mathbf{x}\,(t;\ t_0,\,\mathbf{x}_0)\,\| < b^{-1}\,(a\,(\varepsilon))$ (and, therefore, $\|\,\mathbf{y}\,(t;\ t_0,\,\mathbf{x}_0)\,\| < b^{-1}\,(a\,(\varepsilon)))$ for all $t \in [t_0,\,T]$. Let us show that $\|\,\mathbf{y}\,(t;\ t_0,\,\mathbf{x}_0)\,\| < \varepsilon$ for all $t \geqslant t_0$ if only $t_0 \geqslant 0$, $\|\,\mathbf{x}_0\,\| < \delta$. By the choice of number $\delta\,(\varepsilon)$, to do this it is enough to prove that from $t_0 \geqslant T\,(\varepsilon)$, $\|\,\mathbf{y}_0\,\| < b^{-1}\,(a\,(\varepsilon))$ follows $\|\,\mathbf{y}\,(t;\,t_0,\,\mathbf{x}_0)\,\| < \varepsilon$ for all $t \geqslant t_0$.

Let $t_0 \geqslant T\,(\varepsilon)$, $\|\,\mathbf{y}_0\,\| < b^{-1}\,(a\,(\varepsilon))$; then, according to (1.6), $V\,(t_0,\,\mathbf{x}_0) < b\,(b^{-1}\,(a\,(\varepsilon))) = a\,(\varepsilon)$. Let us show that

$$V\,(t,\,\mathbf{x}\,(t;\,t_0,\,\mathbf{x}_0)) < a\,(\varepsilon) \qquad \text{for all} \qquad t \geqslant t_0 \tag{1.9}$$

We assume, to the contrary, that $V\,(t,\,\mathbf{x}\,(t;\,t_0,\,\mathbf{x}_0)) < a\,(\varepsilon)$ when $t \in [t_0,\,t_1)$, but $V\,(t_1,\,\mathbf{x}\,(t_1;\,t_0,\,\mathbf{x}_0)) = a\,(\varepsilon)$, and, consequently, $V^{\cdot}\,(t_1,\,\mathbf{x}\,(t_1;\,t_0,\,\mathbf{x}_0)) \geqslant 0$. According to (1.6), $b^{-1}\,(a\,(\varepsilon)) \leqslant \|\,\mathbf{y}\,(t_1;\,t_0,\,\mathbf{x}_0)\,\| \leqslant \varepsilon$ and since $t_1 > T\,(\varepsilon)$, $V^{\cdot}\,(t_1,\,\mathbf{x}\,(t_1;\,t_0,\,\mathbf{x}_0)) < 0$, which leads to a contradiction. On the basis of (1.6), from (1.9) it follows that $\|\,\mathbf{y}\,(t;\ t_0,\,\mathbf{x}_0)\,\| < \varepsilon$ for all $t \geqslant t_0$.

N o t e. We have in fact proved that for any $\varepsilon > 0$ we can find $T\,(\varepsilon) > 0$ such that

$$V^{\cdot}\,(t,\,\mathbf{x})\,|_{V(t,\,\mathbf{x}) = a(\varepsilon)} < 0 \qquad \text{for all} \qquad t \geqslant T \tag{1.10}$$

2) Let us show that the motion $\mathbf{x} = 0$ is uniformly $\mathbf{y}$-attracting, i.e., with a specified $\delta\,(\varepsilon) > 0$, for any $\alpha \in (0,\,\delta)$ there exists $\tau\,(\alpha) > 0$ such that from $t_0 \geqslant 0$, $\|\,\mathbf{x}_0\,\| < \delta$ follows $\|\,\mathbf{y}\,(t;\ t_0,\,\mathbf{x}_0)\,\| < \alpha$ for all $t \geqslant t_0 + \tau\,(\alpha)$. Let $\alpha \in (0,\,\delta)$ be given; by hypothesis (see (1.8) and (1.7)) $T\,(\alpha) > 0$ exists such that for $t \geqslant T\,(\alpha)$ and $b^{-1}\,(a\,(\alpha)) \leqslant \|\,\mathbf{y}\,\| \leqslant \varepsilon$ ($\alpha < \delta\,(\varepsilon) < b^{-1}\,(a\,(\varepsilon)) < \varepsilon$) we have

$$V^{\cdot}\,(t,\,\mathbf{x}) \leqslant -{}^1/_2 c\,(b^{-1}\,(a\,(\alpha))) \tag{1.11}$$

and, consequently,

$$V^{\cdot}(t, \mathbf{x})|_{V(t,\,\mathbf{x})=a(\alpha)} < 0 \quad \text{for} \quad t \geqslant T(\alpha) \tag{1.12}$$

We set $t_0' = t_0'(\alpha) = \max\{t_0, T(\alpha)\}$, $\tau_1(\alpha) = (2b(\varepsilon) - a(\alpha)) / c(b^{-1}(a(\alpha)))$. Let us show that an instant $t_* \in (t_0', t_0' + \tau_1(\alpha))$ exists for which

$$V(t_*, \mathbf{x}(t_*; t_0, \mathbf{x}_0)) < a(\alpha) \tag{1.13}$$

We assume to the contrary that the inequality $V(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \geqslant a(\alpha)$ holds for all $t \in (t_0', t_0' + \tau_1(\alpha))$. Then on this time interval $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| \geqslant b^{-1}(a(\alpha))$, and, consequently, by virtue of (1.11), $V^{\cdot}(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \leqslant -{}^1/_2 c(b^{-1}(a(\alpha)))$, and from the relation

$$V(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) = V(t_0, \mathbf{x}_0) + \int_{t_0}^{t} V(\xi, \mathbf{x}(\xi; t_0, \mathbf{x}_0))\, d\xi$$

follows

$$\begin{aligned} 0 < a(\alpha) \leqslant V(t_0' + \tau_1(\alpha), \mathbf{x}(t_0' + \tau_1(\alpha); t_0, \mathbf{x}_0)) = \\ V(t_0', \mathbf{x}(t_0'; t_0, \mathbf{x}_0)) + \int_{t_0'}^{t_0'+\tau_1(\alpha)} V^{\cdot}(\xi, \mathbf{x}(\xi; t_0, \mathbf{x}_0))\, d\xi \leqslant \\ b(\varepsilon) - {}^1/_2 c(b^{-1}(a(\alpha)))\tau_1(\alpha) = {}^1/_2 a(\alpha) \end{aligned}$$

which is impossible. On the basis of (1.12) we conclude from (1.13) that $V(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) < a(\alpha)$ for all $t \geqslant t_*$ and, therefore, $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| < \alpha$ for $t \geqslant t_*$. Consequently, $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| < \alpha$ for any $t \geqslant t_0 + \tau(\alpha) > t_*$, where $\tau(\alpha) = T(\alpha) + \tau_1(\alpha)$. The theorem has been proved.

In particular, let $m = n$; then we have the valid

T h e o r e m 2. For the asymptotic stability, uniform in $\{t_0, \mathbf{x}_0\}$, of motion $\mathbf{x} = 0$ it is sufficient, and under the condition that the right hand sides of system (1.1) and their partial derivatives with respect to the coordinates are continuous and bounded in domain (1.2) also necessary, that there exist two positive-definite functions $V(t, \mathbf{x})$ and $W(t, \mathbf{x})$, the first of which admits of an infinitesimal upper bound, and that relation (1.3) be fulfilled for any λ and μ, $0 < \lambda < \mu < H$.

P r o o f. The sufficiency follows from Theorem 1. Let us prove the necessity. Under the conditions imposed on the right hand sides of system (1.1) there exists, as Malkin showed [8, 9], a positive-definite function $V(t, \mathbf{x})$ admitting of an infinitesimal upper bound and having a negative-definite derivative $V^{\cdot}(t, \mathbf{x})$. Having set $W(t, \mathbf{x}) \equiv -V^{\cdot}(t, \mathbf{x})$, we obtain two functions satisfying the hypotheses of Theorem 2. Q. E. D.

Together with system (1.1) we consider the "perturbed" system

$$\mathbf{x}^{*\cdot} = \mathbf{X}(t, \mathbf{x}^*) + \mathbf{R}(t, \mathbf{x}^*), \quad \mathbf{R}(t, 0) \equiv 0 \tag{1.14}$$

relative to which we assume the fulfilment of conditions a), b) and (1.5).

T h e o r e m 3. Assume that a function $V(t, \mathbf{x})$ exists, satisfying inequalities (1.6), whose derivative relative to system (1.1) $V^{\cdot}(t, \mathbf{x}) \leqslant -c(\|\mathbf{y}\|)$, and that the condition

$$\frac{\partial V(t, \mathbf{x})}{\partial \mathbf{x}} \cdot \mathbf{R}(t, \mathbf{x})_{\lambda \leqslant \|\mathbf{y}\| \leqslant \mu,\ 0 \leqslant \|\mathbf{z}\| < \infty} \rightrightarrows 0 \quad \text{as} \quad t \to \infty \tag{1.15}$$

is fulfilled for any λ and μ such that $0 < \lambda < \mu < H$. Then the motion $\mathbf{x}^* = 0$ of system (1.14) is asymptotically $\mathbf{y}^*$-stable uniformly in $\{t_0, \mathbf{x}_0^*\}$.

The statement follows from Theorem 1 because the derivatives of function $V(t, \mathbf{x})$ relative to system (1.1) and (1.4), denoted $V_{(1)}^{\cdot}(t, \mathbf{x})$ and $V_{(2)}^{\cdot}(t, \mathbf{x})$, respectively, are connected by the equality

$$\dot{V}_{(2)}(t, \mathbf{x}) = \dot{V}_{(1)}(t, \mathbf{x}) + \frac{\partial V(t, \mathbf{x})}{\partial \mathbf{x}} \cdot \mathbf{R}(t, \mathbf{x})$$

If function $V(t, \mathbf{x})$ has bounded partial derivatives $\partial V / \partial \mathbf{x}$, then, obviously, condition (1.5) can be replaced by

$$\mathbf{R}(t, \mathbf{x})_{\lambda \leqslant \|\mathbf{y}\| \leqslant \mu,\ 0 \leqslant \|\mathbf{z}\| < \infty} \rightrightarrows 0 \quad \text{as} \quad t \to \infty$$

2. We consider the controlled system

$$\mathbf{x}^{\cdot} = \mathbf{X}(t, \mathbf{x}, \mathbf{u}), \quad \mathbf{u} \in \mathbf{R}^r \tag{2.1}$$

whose control performance index is understood as the condition of minimum of the integral [10]

$$J = \int_{t_0}^{\infty} \omega(t, \mathbf{x}[t], \mathbf{u}[t])\, dt, \quad \omega \geqslant 0$$

Controls $\mathbf{u}(t, \mathbf{x})$, continuous in domain (1.4), for which system (2.1) with $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ satisfies conditions a) and b) from Sect. 1, are looked upon as admissible controls. If a certain class $K = \{\mathbf{u}(t, \mathbf{x})\}$ of admissible controls $\mathbf{u}(t, \mathbf{x})$ has been chosen, we speak of optimal $\mathbf{y}$-stabilization in class K [11]; since the case being examined class K coincides with the whole set of admissible controls, we speak of optimal $\mathbf{y}$-stabilization, omitting the words "in class K". Following [10], we adopt the notation

$$B[V, t, \mathbf{x}, \mathbf{u}] = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \mathbf{x}} \cdot \mathbf{X}(t, \mathbf{x}, \mathbf{u}) + \omega(t, \mathbf{x}, \mathbf{u})$$

Theorem 4. Assume that functions $V(t, \mathbf{x})$, $W(t, \mathbf{x})$ and $\mathbf{u}^\circ(t, \mathbf{x})$ exist, possessing the following properties:

1) for any $T > 0$ there exits $L(T) > 0$ such that the condition

$$\| \mathbf{X}(t, \mathbf{x}', \mathbf{u}^\circ(t, \mathbf{x}')) - \mathbf{X}(t, \mathbf{x}'', \mathbf{u}^\circ(t, \mathbf{x}'')) \| \leqslant L \| \mathbf{x}' - \mathbf{x}'' \|$$

is fulfilled in domain $0 \leqslant t \leqslant T$, $\|\mathbf{x}\| \leqslant H$;

2) function $V(t, \mathbf{x})$ satisfies inequalities (1.6), while $W(t, \mathbf{x})$ satisfies inequality (1.7);

3) the relation

$$-\omega(t, \mathbf{x}, \mathbf{u}^\circ(t, \mathbf{x})) + W(t, \mathbf{x})_{\lambda \leqslant \|\mathbf{y}\| \leqslant \mu,\ 0 \leqslant \|\mathbf{z}\| < \infty} \rightrightarrows 0 \quad \text{as} \quad t \to \infty$$

is valid for any λ and μ; $0 < \lambda < \mu < H$;

4) $B[V, t, \mathbf{x}, \mathbf{u}^\circ(t, \mathbf{x})] = 0$;

5) $B[V, t, \mathbf{x}, \mathbf{u}] \geqslant 0$ for any $\mathbf{u}$.
Then function $\mathbf{u} = \mathbf{u}^\circ(t, \mathbf{x})$ solves the optimal $\mathbf{y}$ -stabilization problem.

Proof. By virtue of 1) — 3) the functions $V(t, \mathbf{x})$ and $W(t, \mathbf{x})$ satisfy the hypotheses of Theorem 1 for system

$$\mathbf{x}^\cdot = \mathbf{X}(t, \mathbf{x}, \mathbf{u}^\circ(t, \mathbf{x}))$$

and, consequently, the motion $\mathbf{x} = 0$ of this system is asymptotically $\mathbf{y}$ -stable uniformly in $\{t_0, \mathbf{x}_0\}$. Let $\mathbf{u}^*(t, \mathbf{x})$ be some admissible control ensuring the asymptotic $\mathbf{y}$ -stability of the motion $\mathbf{x} = 0$ of system (2.1). By virtue of the second of inequalities (1.6)

$$\lim_{t\to\infty} V(t, \mathbf{x}^\circ[t]) = \lim_{t\to\infty} V(t, \mathbf{x}^*[t]) = 0$$

Hence, according to [11], follows the result required.

In analogous fashion, using the results in [11] and the many well-known theorems on asymptotic $\mathbf{y}$ -stability, we can obtain a number of optimal $\mathbf{y}$ -stabilization criteria, just as Theorem 4 was obtained from Theorem 1.

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